Fermions as extra dimensions

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1 The setup

For a massless fermionic field ψ, the standard Lagrangian is $\imath\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi$. In anology to the case of vectors, the idea is to embed the 4-dimensional "field manifold" $\mathscr{M}$ derived from $\psi(x^0,x^1,x^2,x^3)$ in a 5-dimensional space, whith the 4-volume of $\mathscr{M}$ serving as the action up to a multiplicative constant. For this sake, it is necessary to estimate the induced metric m by means of Gauss' formula $m_{\mu\nu} = \Gamma_{ab}\frac{\partial X^a \partial X^b}{\partial x^\mu \partial x^\nu}$, where Γ is the metric of the embedding space, the X are the five contravariant coordinates at the positon of the field manifold and the x are the internal coordinates simply identified with parts of the X and interpreted as the spacetime coordinates. However, since ψ shall become a field living on spacetime, Gauss' formula shall be modified such that in place of the partial derivative the covariant derivative for spin 1/2 fields acts on ψ. The so modified result shall be denoted $m_{\mu\nu}$.

The appropriate embedding space metric Γ reads, denoted in terms of the line element

$$\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu \mathbb{1}_4 + G^2\hbar^2\bar{\psi}\gamma_\mu(\mathrm{d}x^\mu \mathrm{d}\psi + \mathrm{d}\psi \mathrm{d}x^\mu)\,. \tag{1}$$

$\gamma_\mu = \eta_{ab}e^a_\mu\gamma^b$, where η is the Minkowski metric, e are the tetrads, γ are the Dirac gamma matrices, G is Newton's constant and $\hbar$ is Planck's reduced constant, respectively. The dimension of ψ is $length^{-3/2}$. The conversion factor could involve some numeric factor order of magnitude unity.

2 Estimating the induced metric and Lagrangian

For the sake of better readability, the factor ℓ_{pl}^4 is supressed throughout most parts of this section - quasi Planckian units are used - and will only be reintroduced in the final formulas. Furthermore the factor $\mathbb{1}_4$, which would multiply g, is suppressed. The contravariant X^i are grouped as a 5-vector in embedding space as $\mathrm{col}(x^0\ x^1\ x^2\ x^3\ \psi)$. Differentiation w.r.t. the x^μ yields the four 5-vectors

$$\begin{array}{c}\mathrm{col}(1\ 0\ 0\ 0\ \psi_{|0})\\ \mathrm{col}(0\ 1\ 0\ 0\ \psi_{|1})\\ \mathrm{col}(0\ 0\ 1\ 0\ \psi_{|2})\\ \mathrm{col}(0\ 0\ 0\ 1\ \psi_{|3})\\ ,\end{array} \tag{2}$$

where the pipe symbolizes the covariant derivative for spin 1/2 fields. To get the 4 covariant 5-vectors, the above have to be multiplied by

$$\Gamma = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} & \bar\psi\gamma_0 \\ g_{10} & g_{11} & g_{12} & g_{13} & \bar\psi\gamma_1 \\ g_{20} & g_{21} & g_{22} & g_{23} & \bar\psi\gamma_2 \\ g_{30} & g_{31} & g_{32} & g_{33} & \bar\psi\gamma_3 \\ \bar\psi\gamma_0 & \bar\psi\gamma_1 & \bar\psi\gamma_2 & \bar\psi\gamma_3 & 0 \end{pmatrix}. \tag{3}$$

This yields

$$\begin{pmatrix} g_{00}+\bar\psi\gamma_0\psi_{|0} \\ g_{10}+\bar\psi\gamma_1\psi_{|0} \\ g_{20}+\bar\psi\gamma_2\psi_{|0} \\ g_{30}+\bar\psi\gamma_3\psi_{|0} \\ \bar\psi\gamma_0 \end{pmatrix} \begin{pmatrix} g_{01}+\bar\psi\gamma_0\psi_{|1} \\ g_{11}+\bar\psi\gamma_1\psi_{|1} \\ g_{21}+\bar\psi\gamma_2\psi_{|1} \\ g_{31}+\bar\psi\gamma_3\psi_{|1} \\ \bar\psi\gamma_1 \end{pmatrix} \begin{pmatrix} g_{02}+\bar\psi\gamma_0\psi_{|2} \\ g_{12}+\bar\psi\gamma_1\psi_{|2} \\ g_{22}+\bar\psi\gamma_2\psi_{|2} \\ g_{32}+\bar\psi\gamma_3\psi_{|2} \\ \bar\psi\gamma_2 \end{pmatrix} \begin{pmatrix} g_{03}+\bar\psi\gamma_0\psi_{|3} \\ g_{13}+\bar\psi\gamma_1\psi_{|3} \\ g_{23}+\bar\psi\gamma_2\psi_{|3} \\ g_{33}+\bar\psi\gamma_3\psi_{|3} \\ \bar\psi\gamma_3 \end{pmatrix}. \tag{4}$$

To arrive at the induced metric, the (transposes of the) 4 contravariant vectors (equation (2)) have to be multiplied by the 4 covariant ones (equation (4)). Hence, the induced metric based on covariant derivatives reads

$$m = \begin{pmatrix} g_{00}+2\bar\psi\gamma_0\psi_{|0} & g_{01}+\bar\psi\gamma_0\psi_{|1}+\bar\psi\gamma_1\psi_{|0} & g_{02}+\bar\psi\gamma_0\psi_{|2}+\bar\psi\gamma_2\psi_{|0} & g_{03}+\bar\psi\gamma_0\psi_{|3}+\bar\psi\gamma_3\psi_{|0} \\ g_{10}+\bar\psi\gamma_0\psi_{|1}+\bar\psi\gamma_1\psi_{|0} & g_{11}+2\bar\psi\gamma_1\psi_{|1} & g_{12}+\bar\psi\gamma_1\psi_{|2}+\bar\psi\gamma_2\psi_{|1} & g_{13}+\bar\psi\gamma_1\psi_{|3}+\bar\psi\gamma_3\psi_{|1} \\ g_{20}+\bar\psi\gamma_0\psi_{|2}+\bar\psi\gamma_2\psi_{|0} & g_{21}+\bar\psi\gamma_1\psi_{|2}+\bar\psi\gamma_2\psi_{|1} & g_{22}+2\bar\psi\gamma_2\psi_{|2} & g_{23}+\bar\psi\gamma_2\psi_{|3}+\bar\psi\gamma_3\psi_{|2} \\ g_{30}+\bar\psi\gamma_0\psi_{|3}+\bar\psi\gamma_3\psi_{|0} & g_{31}+\bar\psi\gamma_1\psi_{|3}+\bar\psi\gamma_3\psi_{|1} & g_{32}+\bar\psi\gamma_2\psi_{|3}+\bar\psi\gamma_3\psi_{|2} & g_{33}+2\bar\psi\gamma_3\psi_{|3} \end{pmatrix}. \tag{5}$$

One can straightforwardly use the approximation for the determinant of a matrix plus a small pertubation, where only the trace of the latter remains:

$$\det m \approx (1 + 2\ell_{pl}^4 \bar{\psi}\gamma^\mu \psi_{|\mu}) \det g \,. \tag{6}$$

Its square root minus the cosmological term, multiplied by the appropriate forefactor, then approximately is

$$\frac{1}{\ell_{pl}^4}\left(\sqrt{\det m} - \sqrt{\det g}\right) \approx \imath \bar{\psi}\gamma^\mu \psi_{|\mu} \sqrt{-\det g}\,, \tag{7}$$

what exactly reproduces the Dirac Lagrangian. The $\imath$ comes in, since $\det m$ is negative.

3 A new variant of Supersymmetry?

In line element (1) part of the degrees of freedom of the embedding space can be addressed as spacetime-like, while the fifth is field-like, namely fermionic. Bosonic fields are even simpler to add to arrive at an further extended embedding space. For vectors this leads to a geometrization of the Born-Infeld theory [1]. For scalars the situation is quite straightforward [2]. Here pars pro toto, complex scalar fields Φ with dimension $mass^{-1/2}length^{-3/2}$ shall be added. Then the line element in embedding space reads

$$\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu \mathbb{1}_4 + G^2\hbar^2 \bar{\psi}\gamma_\mu(\mathrm{d}x^\mu \mathrm{d}\psi + \mathrm{d}\psi \mathrm{d}x^\mu) + G^2\hbar^3(\mathrm{d}\Phi\mathrm{d}\Phi^* + \mathrm{d}\Phi^*\mathrm{d}\Phi)\mathbb{1}_4\,, \tag{8}$$

where the asterix means complex conjugation.

So one arrives at a space spanned by bosonic as well as fermionic degrees of freedom, together with those of spacetime. This could allow a new glance at the topic of sumersymmetry.

References

[1] Vones 1

[2] Vones 2