Born-Infeld Theory Geometrized

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Abstract

A symmetry facilitated by Newton's constant inside a generalized metric brings together the concepts of spacetime and vector fields in a way reminding of the unification of space and time by Special Relativity, with a subtle difference though. This can be regarded as a geometrization of the historic "New Field Theory" of Born and Infeld. The underlying symmetry allows to comprehend why the vacuum does not gravitate.

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1 Introduction and history

The nonrelativistic Lagrangian of a free object reads $\frac{m}{2}\dot{\vec{x}}^2$, where m is mass, $\vec{x}$ is position in 3-space, and the overdot means derivative w.r.t. coordinate time t. Special Relativity (SR) taught that this is just a low velocity approximation of the correct expression with the rest term explicitly omitted, $mc^2\left[1-\sqrt{1-\left(\frac{\dot{\vec{x}}}{c}\right)^2}\right]$. c is the velocity of light facilitating the Lorentz symmetry of spacetime. It will be set unity in the following except where explicitly stated otherwise.

It is a nearlying conjecture that there is an escalation for the case of a massless free Abelian vector field Lagrangian $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is derived from the vector field A (in particular, the electromagnetic field) and indices everywhere in this paper are moved by means of the spacetime metric $g_{\mu\nu}$ whose determinant will be denoted $\det\left(g_{\mu\nu}\right)$. It is helpful to reckognize that this term has no individual forefactor comparable to mass in Special Relativity, rather this is absorbed in the field per definition. As will turn out later, such individual factors exist and discriminate among different vector fields.

Intriguingly, the correct result was published already in 1934 by born Born and Infeld [1]. However, as will be argued in the following, they did not reckognize the underlying symmetry. If they had, some aspects of field physics might have developed differently, and definitely the associated millenium problem would not read like it does.

The final version of Born and Infeld for the electromagnetic field action was

$$S=b^2\int\left[\sqrt{-\det\left(g_{\mu\nu}\right)}-\sqrt{-\det\left(g_{\mu\nu}+\frac{1}{b}F_{\mu\nu}\right)}\right]\mathrm{d}^4x=$$
$$=b^2\int\left[1-\sqrt{1+\frac{1}{2b^2}F_{\mu\nu}F^{\mu\nu}-\frac{1}{16b^4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2}\right]\sqrt{-\det\left(g_{\mu\nu}\right)}\,\mathrm{d}^4x\,, \quad (1)$$

where b is a parameter with the same dimension as F, that is the square root from an energy density. $\tilde{F}$ is the Hodge dual of F. The notation and conventions of [2] are applied. This clearly implies nonlinear field equations, but for a completly different reason than the internal gauge groups which are known today in the context of Yang-Mills fields.

From the second line of equation (1) it is obvious that the desired approximation works if b^2 is sufficiently large. [1] conjectured $|b|$ to be order of magnitude $\frac{m_-^2}{|e|^3}$, where m_- is the electron mass and e is the elementary charge, since their considerations focused on the field energy of the electron. This was a quite straightforward idea in those early days of particle physics. But today, this is falsified from both theory and observation. There is a vast variety of elementary particles and picking the electron appears as quite random. The limits from observation recently were estimated by [3, 4]. Rather, one should acknowledge that the expansion of the square root in the second line of equation (1) yields a dominant contribution $-b^2\sqrt{-\det g}\,\mathrm{d}^4x$, which acts as cosmological term. It is derived on a "classical" route, furthermore it is omitted to be in line with observation. Nevertheless, it has to be consistent with the ideas of quantum physics which specifies the vacuum by a cutoff of momentum integrals at Planckian energies. So b should be of Planckian orders of magnitude.

2 The geometric framework

[1] deals with electrodynamics what yet is sufficient to comprehend the mechanism. The comparable situation in SR would be that there existed one single concrete object living in abstract spacetime. Like there the action of the object is derived from the embedding of the 1-dimensional worldline in 4-dimensional spacetime, here the action of electrodynamics will be derived from the embedding of a 4-dimensional "field manifold" called $\mathscr{M}$ in an 8-dimensional embedding space. This embedding space is spanned by 4-spacetime x^μ and the abstract vector field $\mathscr{A}^\mu$ measured in energy units as it its well established in high energy physics. Its (generalized) metric is the tensor product

$$\Gamma = g \otimes \begin{pmatrix} 1 & -G \\ G & 0 \end{pmatrix} , \tag{2}$$

where G is Newton's constant with dimension $\frac{length}{momentum}$. The signs of the off-diagonal elements could be flipped, what means a change of the sign of charge and has no influence on the result. Written out explicitly, the line element squared reads (both possible relative signs indicated; sign convention of g can be left unspecified)

$$\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu \pm G\, \mathrm{d}x_\mu \wedge \mathrm{d}\mathscr{A}^\mu , \tag{3}$$

and has units of $length^2$. That G appears is the consequence of the Planckian scale. Like c acts as a metric conversion factor in SR, so G now does at the level of field physics. Spacetime and vector fields living on it are unified like space and time are unified in SR. This is the central innovation brought forward by this paper, which up to now has not been introduced anywhere, neither constructively nor in the context of no-go theorems.

Whereas Born and Infeld focused on invariancies inside spacetime alone, now the field action can be derived from pure geometry. The 4 field equations $\mathscr{A}^{\mu}(x^0,x^1,x^2,x^3)$ define the 4-dimensional $\mathscr{M}$ embedded in the 8-dimensional space. Its 4-volume, which is the action apart from an overall constant, can be derived by estimating the induced metric called γ by means of Gauss' formula, with result

$$\gamma_{\mu\nu} = \Gamma_{ij}\frac{\partial X^i}{\partial x^\mu}\frac{\partial X^j}{\partial x^\nu} = g_{\mu\nu} \pm G(\partial_\mu \mathscr{A}_\nu - \partial_\nu \mathscr{A}_\mu) \,, \tag{4}$$

where X are the eight contravariant embedding space coordinates at the location of $\mathscr{M}$, and x are the four contravariant coordinates on $\mathscr{M}$ simply identified with part of the X. ∂ is a partial derivative. The mechanism explains where the derivative terms in the action originate from in terms of differential geometry.

The 4-volume is proportional to the determinant of the absolute value of the induced metric. Hence, equation (4) immediately implies the first line of equation (1) if, for the concrete case of electrodynamics the abstract vector field is replaced by $\mathscr{A} \to eA$, where the factors bear their historic units of $\sqrt{momentum \cdot length}$ and $\sqrt{\frac{momentum}{length}}$, respectively, and

$$|b| \to \frac{1}{|e|G} \tag{5}$$

(remember what was said about the individual constant earlier). b^2 is the Planckian energy density $\frac{1}{\hbar G^2}$ divided by the fine structure constant α understood as the low energy value. This value originates from those conjectured in [1] by the substitution $\left(\frac{e}{m_-}\right)^2 \to G$.

One might challenge the claim of α being involved as an individual constant. The argument is: if there was an object carrying an elementary electric charge e inserted in the electromagnetic field, it would aquire an additional momentum of eA. Though fictional, this is the only quantity around which actually has dimension *momentum* in contrast to the vector field, whose natural dimension rather is $\sqrt{\frac{1}{momentum \cdot length^3}}$, which is multiplied by the dimension of $\hbar$ in electrodynamics and by the dimension of $\sqrt{\hbar^3}$ in present day field physics, respectively. $\hbar$ is Planck's constant.

3 Yang-Mills fields

As meanwhile is known, the electromagnetic field is not even a fundamental one, rather lies in a specific manner inside the SU(2)xU(1) symmetry. The fundamental fields which are all massless prior to the Higgs mechanism are 1 weak hypercharge, 3 weak isospin and 8 gluons. Like in SR there can live numerous objects in one and olny spacetime, here live a dozen of vector fields in the one and only embedding space, each with its own $\mathscr{M}_n$ with n running from 1 to 12.

The currently used action for such fields is (the index of the field denoted as upper index) with a running through a subset of the n

$$L = -\frac{1}{4}\sum_a F^a_{\mu\nu} F^{a\,\mu\nu} \qquad \text{with} \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \tilde{\alpha} f^{abc} A^b_\mu A^c_\nu \,, \tag{6}$$

where $\tilde{\alpha}$ is the respective coupling constant. f are the totally antisymmetric structure constants of the internal gauge group, so this term is also antisymmetric in μ and ν. The parts of F containing partial derivatives are identic to those discussed in the context of electrodynamics, and their geometrization comes along exactly in the same way only with $\tilde{\alpha}$ replacing α. The part contaning the f can be regarded as originating from an interaction among the respective field manifolds, what is not a topic of this paper. But the term can be taken into account when equation (1) is evaluated. The Higgs mechanism then is another story not be regarded here.

The antisymmetry of the dynamic parts (those containing derivatives) of the action is woven into equation (2), and this makes sense. This has directly to do with the spacetime properties of any vector field, projecting out the pure helicity 1 component. This would not work if the lower right entrance of the second factor of Γ was unequal zero. In particular, the natural supression of this entrance by a factor of G^2 would lead to diagonal contributions to the determinant same order of magnitude as the off-diagonal ones.

One aspect could turn out as quite essential. Any $\mathscr{M}_n$ is a membrane, with the most simple action resulting from embedding associated. Mathematically, string and membrane theory yet has reached an enormously advanced stage, but never has a membrane been given the physical interpretation as it is the case here.

4 Omitting the vacuum

From equation (2) emerges an argument why the cosmological term can be omitted, in contrast to the rest mass term in SR. The second factor of Γ does not describe Lorentz symmetry. Rather, introducing the transformation matrix $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its transpose K^T (symbols not to be confused with similar ones used elsewhere), it is

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} 1 & -G \\ G & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & ab - Gad + Gbc \\ ab - Gbc + Gad & b^2 \end{pmatrix} . \tag{7}$$

If this factor of the metric shall be conserved by this transformation, the three restricting equations are $b = 0$, $\quad a = d = \pm 1$. This leaves c as a free parameter. The determinant of the transformation matrix is unity in any case. If the positive sign is chosen for a, such transformation applied to a vector has the effect $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} x \\ \mathscr{A} \end{pmatrix} = \begin{pmatrix} x \\ \mathscr{A} + cx \end{pmatrix}$, where c has the inverse dimension of Newton's constant. This means only $\mathscr{A}$ is modified. In contrast, space is invariant under this transformation thus defining an absolute frame of reference. This is the crucial difference to Lorentz symmetry and can be seen as the reason why the (vector) vacuum does not gravitate. To see how this can be extended to locally Minkowskian spacetime, it is sufficient to regard time plus one spatial dimension, i.e.an embedding space with metric $\mathrm{diag}(1, -1) \otimes \begin{pmatrix} 1 & -G \\ G & 0 \end{pmatrix}$. The transformation matrix $K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ c_1 & 1 & C & 0 \\ 0 & 0 & 1 & 0 \\ -C & 0 & c_2 & 1 \end{pmatrix}$, where c_1, c_2, C are free parameters, leaves this metric invariant. c_1 and c_2 are just the parameters of transformations inside the respective pair of x and p discussed above, whereas C mixes different degrees of freedom of spacetime. Taking account of all degrees of freedom of spacetime, the result are gauge transformations $\mathscr{A}^\mu \to \mathscr{A}^\mu + \frac{\partial \Phi}{\partial x^\mu}$ with quadratic $\Phi = c_{\alpha\beta} x^\alpha x^\beta$, where all the $c_{\alpha\beta}$ are constants.

Omitting the cosmological term is to say, gravity feels the 4-volume of any $\mathscr{M}_n$ minus the 4-volume of the manifold $\mathscr{A}^\mu = const^\mu$, that simply is spacetime itself. Spacetime is intrinsically curved because the embedding space is while its embedding equations are trivial, whereas any $\mathscr{M}_n$ is intrinsically curved for both reasons, the curved embedding space and the nontrivial embedding equations. Topological aspects are determined by those of the embedding space.

Having estimated where the geometrized Born-Infeld theory differs from SR, namely what regards the constant (mass there, cosmological here) term, one can summarize the similarities. For this sake, half of the antisymmetrized gradient of $\mathscr{A}$ here shall be identified with velocity there. The found role of the coupling constant is very plausible, since it is the only constant associated with a field other than the mass of its quanta. τ is proper time of any object and V_4 is the 4-volume

theory	mapping	action proportional to	global constant	individual constant
Special Relativity	1 time $\rightarrow$3-position	$\int_{worldline} \mathrm{d}\tau$	c	m
Born-Infeld	4-spacetime $\rightarrow$4-vector field	$\int_{\mathscr{M}} \mathrm{d}V_4$	$\frac{1}{G}$	$\frac{1}{\alpha}$

of any $\mathscr{M}$. α shall mean the respective coupling constant.

References

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