Born-Infeld Theory Geometrized

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Abstract

A symmetry facilitated by Newton's constant inside a generalized metric brings together the concepts of spacetime and vector fields in a way reminding of the unification of space and time by Special Relativity, with a subtle difference though. This can be regarded as a geometrization of the historic "New Field Theory" of Born and Infeld. The underlying symmetry brings some insight in the fact that the cosmological term does not gravitate.

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1 Introduction and history

The nonrelativistic Lagrangian of a free object reads $\frac{m}{2}\dot{\vec{x}}^2$, where m is mass, $\vec{x}$ is position in 3-space, and the overdot means derivative w.r.t. coordinate time t. Special Relativity (SR) taught that this is just a low velocity approximation of the correct expression with the rest term explicitly omitted, $mc^2\left[1-\sqrt{1-\left(\frac{\dot{\vec{x}}}{c}\right)^2}\right]$. c is the velocity of light facilitating the Lorentz symmetry of spacetime. It will be set unity in the following except where explicitly stated otherwise.

It is a nearlying conjecture that there is an escalation for the case of a massless free Abelian vector field Lagrangian $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is derived from the vector field A (in particular, the electromagnetic field) and indices everywhere in this paper are moved by means of the spacetime metric $g_{\mu\nu}$ which shall be abbreviated as g and whose determinant will be denoted $\det\left(g_{\mu\nu}\right)$. It is helpful to reckognize that this term has no individual forefactor comparable to mass in Special Relativity, rather this is absorbed in the field per definition. As will turn out later, such individual factors exist and discriminate among different vector fields.

Intriguingly, the correct result was published already in 1934 by born Born and Infeld [1], apart from the order of magnitude. However, as will be argued in the following, they did not reckognize the underlying symmetry. If they had, some aspects of field physics might have developed differently, although the quantitative effects are very small. But when it comes to fundamental qualitative aspects like the associated millenium problem, then the underlying symmetry is crucial.

The final version of Born and Infeld for the electromagnetic field action was

$$S=b^2\int\left[\sqrt{-\det\left(g_{\mu\nu}\right)}-\sqrt{-\det\left(g_{\mu\nu}+\frac{1}{b}F_{\mu\nu}\right)}\right]\mathrm{d}^4x=$$

$$=b^2\int\left[1-\sqrt{1+\frac{1}{2b^2}F_{\mu\nu}F^{\mu\nu}-\frac{1}{16b^4}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2}\right]\sqrt{-\det\left(g_{\mu\nu}\right)}\,\mathrm{d}^4x\,, \qquad (1)$$

where b is a parameter with the same dimension as F, that is the square root from an energy density. $\tilde{F}$ is the Hodge dual of F. The entire formula is taken from reference [2], whose notation and conventions are applied. This clearly implies nonlinear field equations, but for a completly different reason than the internal gauge groups which are known today in the context of Yang-Mills fields.

It shall be clarified that - despite of the central role Newton's constant will play - this paper is not about gravity. Hence one can replace g by the Minkowski metric η in the above formula and everywhere else without missing the substance. Equation (5) becomes particularly simple then and appendix II is not relevant. But since Born and Infeld introduced this convincing way of coupling vector fields to the metric, this is retained here. It can be added that publications exist dealing with so called Born-Infeld gravity theories where essentially in place of the anti-symmetric F comes the symmetric Ricci tensor, but these are not relevant here.

From the second line of equation (1) it is obvious that the desired approximation works if b^2 is sufficiently large. [1] conjectured $|b|$ to be order of magnitude $\frac{m_-^2}{|e|^3}$, where m_- is the electron mass and e is the elementary charge, since their considerations focused on the field energy of the electron. This was a quite straightforward idea in those early days of particle physics. But today, this is falsified from both theory and observation. There is a vast variety of elementary particles and picking the electron appears as quite random. The limits from observation recently were estimated by [3, 4]. Rather, one should acknowledge that the expansion of the square root in the second line of equation (1) yields a dominant contribution $-b^2\sqrt{-\det g}\,\mathrm{d}^4x$, which acts as cosmological term. It is derived on a "classical" route, furthermore it is omitted to be in line with observation. Nevertheless, it has to be consistent with the ideas of quantum physics which specifies the vacuum by a cutoff of momentum integrals at Planckian energies. So b should be of Planckian orders of magnitude.

In the context of string theory, the Born-Infeld action has gained renewed attention, but nowhere the main idea of this paper is developed, namely that fields span extra dimensions attached to spacetime, which in the case of vectors become paired with the degrees of spacetime much like in phase space.

2 Motivation

The most fundamental symmetries of nature are those mediated by the elements of the Planckian set of units: the velocity of light, Planck's constant $\hbar$ and Newton's constant G. The velocity of light appears inside the line element of spacetime in a very transparent way which has made it the superstar among all constants, even reckognized by the broad public. Statistical physics in union with quantization teaches that physical quantities are to be expressed as pure numbers in a uniquely defined way by means of the phase space volume form $\mathrm{d}\Theta = \pm\frac{\mathrm{d}p\wedge\mathrm{d}x}{\hbar}$, where x is position and p is momentum, respectively. In contrast, the current role of G

is way less comprehensible. It may be worth trying to put it inside some generalized metric like c and $\hbar$. Since it has dimension $\frac{length}{momentum}$ or equivalent - what remarkably only is the case in our number of dimensions - it could well find its place inside some modification of $\mathrm{d}\Theta$. The space of plausible possibilities is quite restricted from dimensional analysis to $(\mathrm{d}x, \mathrm{d}p) \begin{pmatrix} k_1 & -G \\ G & k_2 G^2 \end{pmatrix} \begin{pmatrix} \mathrm{d}x \\ \mathrm{d}p \end{pmatrix}$, where k_1 and k_2 are dimensionless constants and a global forfactor of $\frac{1}{G\hbar}$ has been extracted.

In this paper, such modification will not be discussed in the framework of usual phase space physics. Rather, in current physics there are other quantities to which is assigned the dimension of *momentum* for good reasons, despite of the fact that their natural dimension is quite different. These are bosonic fields whose absolut values squared essentially count the probability density of quanta. As will be demonstrated in the following, this route leads to a geometrization of the Born-Infeld theory if $k_1 = 1$ and $k_2 = 0$ are chosen.

3 The geometric framework

[1] deals with electrodynamics what yet is sufficient to comprehend the mechanism. The comparable situation in SR would be that there existed one single concrete object living in abstract spacetime. Like there the action of the object is derived from the embedding of the 1-dimensional worldline in 4-dimensional spacetime, here the action of electrodynamics will be derived from the embedding of a 4-dimensional “field manifold” called $\mathscr{M}$ in an 8-dimensional embedding space. This embedding space is spanned by 4-spacetime x^μ and the abstract vector field $\mathscr{A}^\mu$ measured in energy units as it is well established in high energy physics. Its (generalized) metric is the tensor product

$$U = g \otimes \begin{pmatrix} 1 & -G \\ G & 0 \end{pmatrix} , \tag{2}$$

where the overall sign is not important. Whether numerical factors order of magnitude unity are involved is not decisively clear and these shall be included in the definition of G. The signs of the off-diagonal elements could be flipped, what means a change of the sign of charge and has no influence on the result. Written out explicitly, the line element squared reads (both possible relative signs indicated; sign convention of g can be left unspecified)

$$\mathrm{d}s^2 = \mathrm{d}x_\mu \mathrm{d}x^\mu \pm G\, \mathrm{d}x_\mu \wedge \mathrm{d}\mathscr{A}^\mu , \tag{3}$$

and has units of $length^2$. That G appears is the consequence of the Planckian scale. Like c acts as a metric conversion factor in SR, so G now does at the level of field physics. Spacetime and vector fields living in it are unified like space (or spatial positions of objects) and time are unified in SR. This is the central innovation brought forward by this paper, which up to now has not been introduced anywhere, neither constructively nor in the context of no-go theorems.

Whereas Born and Infeld focused on invariancies inside spacetime alone, now the field action can be derived from pure geometry. The 4 field equations $\mathscr{A}^\mu(x^0,x^1,x^2,x^3)$ define the 4-dimensional $\mathscr{M}$ embedded in the 8-dimensional space. Its 4-volume can be derived by estimating the induced metric called γ by means of Gauss' formula

$$\gamma_{\mu\nu} = U_{ij}\frac{\partial X^i}{\partial x^\mu}\frac{\partial X^j}{\partial x^\nu}\ , \tag{4}$$

where X are the eight contravariant embedding space coordinates at the location of $\mathscr{M}$, and x are the four contravariant coordinates on $\mathscr{M}$ simply identified with part of the X and interpreted as coordinates in spacetime according to equation (3). ∂ is a partial derivative. This means $\frac{\partial X^\mu}{\partial x^\nu} = \delta^\mu_\nu$ for $\mu,\nu = 0,1,2,3$. For $\mu = 4,5,6,7$ there is the identification $X^\mu = \mathscr{A}^{\mu-4}$, and this shall be interpreted as a vector field living in spacetime. As will be demonstrated in appendix I, the application of formula (4) does not quite lead to the correct result, rather a further skew-symmetric term appears, where the electromagnetic field couples to the gravitational field in a peculiar way. Clearly, this term has to be removed by explicit substraction. As will be demonstrated and justified in appendix II, this is equivalent to modifying Gauss' formula such that the partial derivatives acting on the $\mathscr{A}^\mu$ are replaced by the covariant ones associated with the first factor in equation (2), that is the metric of spacetime. The result is the modified induced metric

$$\tilde{\gamma}_{\mu\nu} = g_{\mu\nu} \pm G(\partial_\mu\mathscr{A}_\nu - \partial_\nu\mathscr{A}_\mu)\ , \tag{5}$$

what is the quantity on which the Born-Infeld theory is based. The mechanism explains where the derivatives in the action originate from in terms of differential geometry. The relevant element of 4-volume is proportional to the square root from the determinant of the absolute value of the modified induced metric $\tilde{\gamma}$. Hence, equation (5) immediately implies the first line of equation (1) if, for the concrete case of electrodynamics the abstract vector field is replaced by $\mathscr{A} \to eA$, where the factors bear their historic units of $\sqrt{momentum \cdot length}$ and $\sqrt{\frac{momentum}{length}}$, respectively, and

$$|b| \to \frac{1}{|e|G}\ . \tag{6}$$

This value originates from those conjectured in [1] by the substitution $\left(\frac{e}{m_-}\right)^2 \rightarrow G$. b^2 is the Planckian energy density ρ_{pl} divided by the fine structure constant α understood as the low energy value

$$b^2 = \frac{1}{G^2 \hbar \alpha} \approx 137\, \rho_{pl} \; , \tag{7}$$

where $\hbar$ is Plancks reduced constant.

4 Yang-Mills fields

As meanwhile is known, the electromagnetic field is not even a fundamental one, rather lies in a specific manner inside the SU(2)xU(1) symmetry. The fundamental fields which are all massless prior to the Higgs mechanism are 1 weak hypercharge, 3 weak isospin and 8 gluons. Like in SR there can live numerous objects in one and only spacetime, here live a dozen of vector fields in the one and only embedding space, each with its own $\mathscr{M}_n$ with n running from 1 to 12.

The currently used action for such fields is (the index of the field denoted as upper index) with a running through a subset of the n

$$L = -\frac{1}{4}\sum_a F^a_{\mu\nu} F^{a\,\mu\nu} \qquad \text{with} \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \tilde{\alpha} f^{abc} A^b_\mu A^c_\nu \; , \tag{8}$$

where $\tilde{\alpha}$ is the respective coupling constant. f are the totally antisymmetric structure constants of the internal gauge group, so this term is also antisymmetric in μ and ν. The parts of F containing partial derivatives are identical to those discussed in the context of electrodynamics, and their geometrization comes along exactly in the same way only with $\tilde{\alpha}$ replacing α in equation (7), again understood as the low energy value. The part containg the f can be regarded as originating from an interaction among the respective field manifolds, what is not a topic of this paper. But the term can be taken into account when equation (1) is evaluated. The Higgs mechanism then is another phenomenon not be regarded here. Despite of the said aspects left aside, the conclusion is clear: The current action used for Yang-Mills fields is not a correct basis for their description, rather modifications in line with the Born-Infeld theory are inevitable.

The antisymmetry of the dynamic parts (those containing derivatives) of the action is woven into equation (2), and this makes sense. This has directly to do with the spacetime properties of any vector field, projecting out the pure helicity

1 component. This would not work if the lower right entrance of the second factor of U was unequal zero. In particular, the natural supression of this entrance by a factor of G^2 would lead to diagonal contributions to the determinant same order of magnitude as the off-diagonal ones.

One aspect could turn out as quite essential. Any $\mathscr{M}_n$ is a membrane, with an action resulting from embedding associated. Mathematically, string and membrane theory yet has reached an enormously advanced stage, but never has a membrane been given the physical interpretation as it is the case here.

5 Omitting the cosmological term

From the coupling to spinor fields it is known that gravity is to be formulated in terms of tetrads, which are four mutually coupled vector fields. This fits well to the ideas presented here, however shall be treated elsewhere because of the amount of discussion necessary. But yet from equation (2) emerges an argument why the cosmological term can be omitted as a source of gravity even if simply the Einstein-Hilbert curvature term is added as reference [1] suggests. The second factor of U does not describe Lorentz symmetry as it is the case in Special Relativity. Rather, introducing the transformation matrix $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and its transpose K^T (symbols not to be confused with similar ones used elsewhere), it is

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 1 & -G \\ G & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & ab - Gad + Gbc \\ ab - Gbc + Gad & b^2 \end{pmatrix} . \tag{9}$$

If this factor of the metric shall be conserved by this transformation, the three restricting equations are $b = 0$, $\quad a = d = \pm 1$. This leaves c as a free parameter. The determinant of the transformation matrix is unity in any case. If the positive sign is chosen for a, such transformation applied to a contravariant vector has the effect $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} x \\ \mathscr{A} \end{pmatrix} = \begin{pmatrix} x \\ \mathscr{A} + cx \end{pmatrix}$, where c has the inverse dimension of Newton's constant. This means only $\mathscr{A}$ is modified, while spacetime remains unchanged thus defining an absolute frame of reference. This is the crucial difference to Lorentz symmetry where no preferred direction of time exists. To see how the first factor of U can be extended to locally Minkowskian spacetime, it is sufficient to regard time plus one spatial dimension, i.e. an embedding space with metric $\mathrm{diag}(1,-1) \otimes \begin{pmatrix} 1 & -G \\ G & 0 \end{pmatrix}$. The transformation matrix

$K_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ c_1 & 1 & C & 0 \\ 0 & 0 & 1 & 0 \\ -C & 0 & c_2 & 1 \end{pmatrix}$, where c_1, c_2, C are free parameters, leaves this metric invariant. c_1 and c_2 are just the parameters of transformations inside the respective pair of x and $\mathscr{A}$ discussed above, whereas C intertwines different degrees of freedom of spacetime.

Omitting the cosmological term is to say, gravity feels the 4-volume (based on $\tilde{\gamma}$) of any $\mathscr{M}_n$ minus the 4-volume of the manifold $\mathscr{A}^\mu = const^\mu$, that simply is spacetime itself. Spacetime is intrinsically curved because the embedding space is while its embedding equations are trivial, whereas any $\mathscr{M}_n$ is intrinsically curved for both reasons, the curved embedding space and the nontrivial embedding equations. Topological aspects are determined by those of the embedding space.

Having estimated where the geometrized Born-Infeld theory differs from SR, namely what regards the constant (mass there, cosmological here) term, one can summarize the similarities. For this sake, half of the antisymmetrized gradient of $\mathscr{A}$ here shall be identified with velocity there. The found role of the coupling constant is very plausible, since it is the only constant associated with a field other than the mass of its quanta. τ is proper time of any object and $\tilde{V}_4$ is the 4-volume

theory	mapping	action proportional to	global constant	individual constant
Special Relativity	1 time $\rightarrow$3-position	$\int_{worldline} \mathrm{d}\tau$	c	m
Born-Infeld	4-spacetime $\rightarrow$4-vector field	$\int_{\mathscr{M}} \mathrm{d}\tilde{V}_4$	$\frac{1}{G}$	$\frac{1}{\alpha}$

of any $\mathscr{M}$, where the tilde shall indicate that it is derived from $\tilde{\gamma}$. α shall mean the respective coupling constant (which, as a matter of definition, can be multiplied by $\hbar$).

6 Acknowledgement

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Appendix I: Calculating the induced metric

Gauss‘formula for the metric induced on an embedded manifold reads

$$\gamma_{\mu\nu} = U_{ij}\frac{\partial X^i}{\partial x^\mu}\frac{\partial X^j}{\partial x^\nu} \ , \tag{10}$$

where the X are the contravariant embedding space coordinates of the embedded manifold, x are contravariant coordinates inside the embedded manifold and U is the metric of the embedding space. Here U is given by equation (2)

$$U = \pm g \otimes \begin{pmatrix} 1 & -G \\ G & 0 \end{pmatrix} \ , \tag{11}$$

where G is Newton's constant with dimension $\frac{length}{momentum}$ and the overall sign is not important. The contravariant X^i are grouped as an 8-vector in embedding space as $\mathrm{col}(x^0 \ x^1 \ x^2 \ x^3 \ \mathscr{A}^0 \ \mathscr{A}^1 \ \mathscr{A}^2 \ \mathscr{A}^3)$. The x are coordinates in spacetime - hence g is the metric of spacetime - and have dimension of *length* while the $\mathscr{A}$ have dimension of *momentum*.

Partial differentiation w.r.t. the x^μ (symbolized by comma) yields the four 8-vectors

$$\begin{array}{l} \mathrm{col}(1 \ 0 \ 0 \ 0 \ \mathscr{A}^0_{,0} \ \mathscr{A}^1_{,0} \ \mathscr{A}^2_{,0} \ \mathscr{A}^3_{,0}) \\ \mathrm{col}(0 \ 1 \ 0 \ 0 \ \mathscr{A}^0_{,1} \ \mathscr{A}^1_{,1} \ \mathscr{A}^2_{,1} \ \mathscr{A}^3_{,1}) \\ \mathrm{col}(0 \ 0 \ 1 \ 0 \ \mathscr{A}^0_{,2} \ \mathscr{A}^1_{,2} \ \mathscr{A}^2_{,2} \ \mathscr{A}^3_{,2}) \\ \mathrm{col}(0 \ 0 \ 0 \ 1 \ \mathscr{A}^0_{,3} \ \mathscr{A}^1_{,3} \ \mathscr{A}^2_{,3} \ \mathscr{A}^3_{,3}) \end{array} \tag{12}$$

To get the 4 covariant 8-vectors, the above have to be multiplied by

$$U = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} & -Gg_{00} & -Gg_{01} & -Gg_{02} & -Gg_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} & -Gg_{10} & -Gg_{11} & -Gg_{12} & -Gg_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} & -Gg_{20} & -Gg_{21} & -Gg_{22} & -Gg_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} & -Gg_{30} & -Gg_{31} & -Gg_{32} & -Gg_{33} \\ Gg_{00} & Gg_{01} & Gg_{02} & Gg_{03} & 0 & 0 & 0 & 0 \\ Gg_{10} & Gg_{11} & Gg_{12} & Gg_{13} & 0 & 0 & 0 & 0 \\ Gg_{20} & Gg_{21} & Gg_{22} & Gg_{23} & 0 & 0 & 0 & 0 \\ Gg_{30} & Gg_{31} & Gg_{32} & Gg_{33} & 0 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

To process the result one has to take into account the relation

$$g_{\mu\alpha}\mathscr{A}^{\alpha}_{,\nu} = (g_{\mu\alpha}\mathscr{A}^{\alpha})_{,\nu} - g_{\mu\alpha\,,\nu}\mathscr{A}^{\alpha} = \mathscr{A}_{\mu,\nu} - g_{\mu\alpha,\nu}\mathscr{A}^{\alpha} \equiv \mathscr{A}_{\mu,\nu} - g_{\alpha\mu,\nu}\mathscr{A}^{\alpha}\ . \tag{14}$$

This yields (only the first two written out, since this is sufficient to see the mechanism)

$$\begin{pmatrix} g_{00} - G\mathscr{A}_{0,0} + Gg_{0\mu,0}\mathscr{A}^{\mu} \\ g_{10} - G\mathscr{A}_{1,0} + Gg_{1\mu,0}\mathscr{A}^{\mu} \\ g_{20} - G\mathscr{A}_{2,0} + Gg_{2\mu,0}\mathscr{A}^{\mu} \\ g_{30} - G\mathscr{A}_{3,0} + Gg_{3\mu,0}\mathscr{A}^{\mu} \\ Gg_{00} \\ Gg_{10} \\ Gg_{20} \\ Gg_{30} \end{pmatrix} \begin{pmatrix} g_{01} - G\mathscr{A}_{0,1} + Gg_{0\mu,1}\mathscr{A}^{\mu} \\ g_{11} - G\mathscr{A}_{1,1} + Gg_{1\mu,1}\mathscr{A}^{\mu} \\ g_{21} - G\mathscr{A}_{2,1} + Gg_{2\mu,1}\mathscr{A}^{\mu} \\ g_{31} - G\mathscr{A}_{3,1} + Gg_{3\mu,1}\mathscr{A}^{\mu} \\ Gg_{01} \\ Gg_{11} \\ Gg_{21} \\ Gg_{31} \end{pmatrix} \tag{15}$$

To arrive at the induced metric, the (transposes of the) 4 contravariant vectors (equation (12)) have to be contracted by the 4 covariant ones (equation (15)). For example zeroth contravariant vector times zeroth covariant vector yields $g_{00} - G\mathscr{A}_{0,0} + Gg_{\alpha 0,0}\mathscr{A}^{\alpha} + G\mathscr{A}_{0,0} - Gg_{\alpha 0,0}\mathscr{A}^{\alpha} = g_{00}$. Zeroth contravariant vector times first covariant vector yields $g_{01} - G\mathscr{A}_{0,1} + Gg_{\alpha 0,1}\mathscr{A}^{\alpha} + G\mathscr{A}_{1,0} - Gg_{\alpha 1,0}\mathscr{A}^{\alpha}$. And so on. So the result is

$$\gamma_{\mu\nu} = g_{\mu\nu} \pm G(\partial_{\mu}\mathscr{A}_{\nu} - \mathscr{A}^{\alpha}\partial_{\mu}g_{\alpha\nu} - \partial_{\nu}\mathscr{A}_{\mu} + \mathscr{A}^{\alpha}\partial_{\nu}g_{\alpha\mu}) = g_{\mu\nu} \pm eG(F_{\mu\nu} + \delta F_{\mu\nu})\ , \tag{16}$$

where F is the usual expression for the electromagnetic field as it appears in particular in the Born-Infeld theory (remember $\mathscr{A} \to eA$), whereas δF is a peculiar extra term.

If one expresses the derivatives of the metric in terms of the Christoffel symbols Γ, one gets

$$\delta F_{\mu\nu} = A^{\alpha}(\Gamma_{\mu\nu\alpha} - \Gamma_{\nu\mu\alpha})\ . \tag{17}$$

In Einsteinian gravity, the Christoffel symbols are symmetric in the last two indices, but not in the first two. So this term vanishes if g is the Minkowski metric, but not in general - rather has to be subtracted explicitely.

Appendix II: "Becoming" a vector Field

To subtract $\delta F_{\mu\nu}$ appears as equally peculiar at first, but eventually is based on a clear reasoning. To comprehend the mechanism, one has to go back to spacetime and Special Relativity. The spatial positions $\vec{x}$ have a double role. At first, together with time t, they are degrees of freedom parametrizing (flat) spacetime. But then there are simple objects living in spacetime - pointlike somethings - breaking the invariances of empty spacetime. For each such object called n comes an $\vec{x}_n$ called the position of the object. Its action is proportional to the length of the worldline $\vec{x}_n(t)$ which is induced from the metric of spacetime. Since the only parameter left is one-dimensional time t, there are no subtleties associated with the transformation properties of positions.

Here, the situation is quite comparable, but not completly. There is an 8-dimensional (intrinsically curved) space where the $\mathscr{A}$ have a double role. At first, together with spacetime x they parametrize this space as $\mathrm{col}(x^0\ x^1\ x^2\ x^3\ \mathscr{A}^0\ \mathscr{A}^1\ \mathscr{A}^2\ \mathscr{A}^3)$. But then, as vector fields live in spacetime, for each such field called n there comes a 4-dimensional field manifold determined by the four embedding equations $\mathscr{A}_n^\mu(x^0,x^1,x^2,x^3)$ with spacetime remaining as the only parameters. When the relevant 4-volume for any such vector field is calculated, its transformation properties in 4-dimensional spacetime have to be taken into account. This means, everywhere in Appendix I the partial derivatives acting on $\mathscr{A}_n$ have to be replaced by the covariant ones (symbolized by semicolon) associated with the spacetime metric. Then the contravariant index is simply lowered in equation (14), i.e. $g_{\mu\alpha}\mathscr{A}^\alpha_{;\nu} = \mathscr{A}_{\mu;\nu}$. Hence δF does not appear any more. In the final result the covariant terms drop out as is well known because of the antisymmetry of F and the symmetry of the Christoffel symbols. So one arrives at equation (5), that is the quantity introduced by [1].

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