A sounder Quantum Algebra

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Abstract

It is argued that the quantum algebra can be derived from SO(3) by contraction in the sense of group theory with well perceivable meaning of the involved parameters.

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Apart from quantum gravity and string theory, the discussions have focused more on interpretations of the historic quantum theory than on modifications. However, it had been Hilbert's immediate criticism that the quantum algebra appears as mathematically rather unsound. Here I shall argue that the still used quantum algebra indeed is only an accurate approximation to the true one.

"Contraction" is a mechanism for the modification of algebra and group structures via singular transformations as discussed in [1]. In the case of SO(3), the starting point is the familar generator algebra (here written in terms of the ladder operators $J_\pm$)

$$[J_+, J_-] = 2J_3 \qquad [J_3, J_\pm] = \pm J_\pm \ . \tag{1}$$

One can apply a coordinate transformation resulting in $h_\pm = a \cdot J_\pm$, consequently the commutators become

$$[h_+, h_-] = 2a^2 J_3 \qquad [J_3, h_\pm] = \pm h_\pm \ . \tag{2}$$

In the singular limit where the pure number a approaches zero, h_+ and h_- commute and the algebra degenerates. Gilmore illustrates the effect in terms of the Little Prince whose outreach relative to the Earth's circumference naturally defines the contraction parameter a. The Little Prince is so tiny that Earth appears as flat to him. This reminds of man in the course of history. But I would like to clarify that in general the human origin of the yardstick is not the central ingredience of a contraction mechanism, as can be seen from the following.

The same reference gives a contraction mechanism from the algebra of U(2) towards the quantum algebra h_4. It is equally transparent in the mathematical sense, but in contrast to the above it lacks interpretation. Conjugate variables are identified with h_+ and h_- above, but neither the meaning of the contraction parameter nor the postulated mixing in of the radial degree of freedom can be understood so far.

It shall now be demonstrated that a contraction from SO(3) is sufficient and well understandable. First, in the above I identify $a^2 = \frac{1}{2j}$, where $j(j+1)$ is the huge eigenvalue of the operator of "spin" squared. j can be integer or half integer, without change of the essence. It can be clarified that the familiar names are used for the involved quantities, but they are not with reference to ordinary space, rather to phase space. In terms of an appropriate basis, $2a^2 J_3 = -I + \frac{1}{j} \cdot diag(2j, 2j-1, ...1, 0) := -I + h_3$, where I is the unit matrix. Second, I assume that the relevant values of the "magnetic" quantum number are low, this means in the second term one is situated close to the right end, so h_3 appears as order of magnitude j^{-1}. Consequently, in the singular limit h_3 becomes invisible in the first commutator. In the other commutators, only h_3 is relevant, since I trivially commutes with all generators. The result is the standard quantum algebra

$$[h_+, h_-] = -I \qquad [h_3, h_\pm] = \pm h_\pm \ , \tag{3}$$

where $h_\pm$ are isomorphic to bosonic creation and annihilation operators, respectively, and h_3 is isomorphic to the particle number operator.

Now one reckognizes that the identity matrix appearing in the first commutator is just the vacuum energy renormalized away. Obviously, this partition of nature into "action" and "energy" does not reflect the true symmetry, where only the entire J_3 appears.

Notation and language in terms of ladder operators were used as a matter of convienience. However, it must be clarified that the above does not refer to fields. Rather it is to be interpreted at the level of first quantization, this means of space(time) itself. Space is bosonic, and the associated phase space is related to the ladder operators in the well known way $h_\pm \leftrightarrow \frac{q \mp \imath p}{\sqrt{2\hbar}}$, with standard meaning of the symbols. The role of time is less straightforward, but this shall not be the topic here, like the implications at the level of fields.

References

[1] Gilmore R, **Lie Groups, Lie Algebras and some of their applications**, John Wiley & Sons, New York (1974)