On the spectrum of time and mass

Gerald Vones

Graz, Austria, European Union
gerald@vones.eu

Abstract

It is argued that the connected component of the quantum gravity algebra is so(4). This means, a 1-dimensional harmonic oscillator is isomorphic to a "Hydrogen atom" in phase space with energy, mass and period being determined from the quantum numbers.

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There are indications that the quantum algebra needs modification. Based on the insights from string theory, reference [1] proposes to replace the unit operator by a momentum dependent expression (see their equation 8). In the following I argue in favor of a much more transparent alternative. For brevity, in the heuristic uncertainty relation $\delta x = \frac{\hbar}{\delta p} + \alpha' \frac{\delta p}{\hbar}$, I call the first term the Heisenberg term and the second the winding term, respectively. Furthermore, I write G in place of $\frac{\alpha'}{\hbar}$ and identify the fundametal length with the only serious candidate available, that is the Planck length. So G essentially is Newton's constant with the velocity of light set unity throughout.

First the focus shall be on the Heisenberg term alone. Hilbert immediately claimed that it is pathologic, and there have been proposed cures for this. Clearly, the symmetry must be "expanded" in the sense of Lie group theory as explained in [2]. It is important to reckognize that the whole process takes place inside the quantum world since Planck's constant is already present in the "contracted" groups, the Heisenberg group and the oscillator group. This means, the contraction parameter - the quantity whose extreme value gives rise to misconceptions - directly involves quantum numbers. It is not yet ultimately clear what is the correct expanded symmetry, however as mentioned in [2] and section 3 of [3], SU(2) is a near-lying candidate. As will become transparent, it straightforwardly allows the desired further expansion.

Locally, SU(2) is equivalent to SO(3), whose algebra has 3 generators fulfilling $[J_i, J_k] = \epsilon_{ijk} J_k$. One can describe a 1-dimensional harmonic oscillator by identifying space and momentum with the generators J_1 and J_2 of the algebra respectively, while energy is associated with J_3. Any manipulation takes place in phase space rather than in ordinary space. $j(j+1)\hbar$, where j is a nonnegative integer (regarding half integer values, see comment later in this note), shall denote the eigenvalue of the operator J^2.

The contraction occurs since our every day experience are phase space rotators with huge j and magnetic quantum numbers around $-j$. This makes the operator of angular momentum projection approximately act as $J_3 \approx -jI$ on all the states under consideration, where I is the unit operator. It shall be emphasized that the situation is even more tricky as if a nonvanishing commutator was contracted to the value zero like for the Galilean group contracted from the Lorentz group. A wrong generator is put in place, namely I for J_3. Since I is to be taken on bord, the contraction formally starts from the u(2) algebra as stated in [2].

Angular momentum projection in phase space is interpreted as the energy of the oscillator very similar to the classical Hamiltonian picture. However, since J_3 is removed from the algebra, the contracted Hamiltonian rather is defined from $J^2 - J_3^2$ and rescaled by a factor of j. Details can be found in [3]. The implications of the entirety of manipulations are grave, in particular what regards the spectrum. In the correct (uncontracted) algebra, the spectrum is bound from both sides.

The rescaled Hamiltonian shall be written as $H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$ with standard meaning of the symbols. Experiment confirms that $\hbar\omega$ is the unit interval of energy. It is the minimal hop in magnetic quantum number which corresponds to the natural number 1. There is only one further parameter available, that is the alternative scale of energy, namely the mass m. Since the contraction parameter is a pure number, it has to involve the quotient of the two scales. The only further quantum number available is j. This suggests the simple identification

$$j(j+1) = \left(\frac{m}{\hbar\omega}\right)^2 \ , \tag{1}$$

where the power of 2 results from the fact that energy (be it J_3, be it $\frac{J^2-J_3^2}{j}$) should run from zero to the mass.

Putting the correct algebra in place of the contracted one unveals that not only the energy interval is quantized, but so is the mass. While this result is already quite impressive, there is still more. Not only the ratio of the scales involved can be relevant, rather their product must be relevant as well. ωm (or its inverse, that is a matter of definition) can be regarded as the eccentricity of the orbit in phase space. Like $\hbar$ is the universal scale of area, G is the universal scale of eccentricity. And like the periodicity of the motion quantizes energy in place of action, it quantizes inverse energy in place of eccentricity. $G\omega$ as the quantum of inverse energy can be regarded as causing the winding term. And like mass compared to the quantum of energy is a contraction parameter, so is inverse mass compared to the quantum of inverse energy, yielding the relevant ratio $\frac{1}{G\omega m}$.

The reckognition that factually a rotator in phase space is to be considered, opens the door for relating this second contraction parameter to a quantum number. As is known from ordinary space, if rotation symmetry is unbroken there is an associated O(4) group in union with the so called Runge-Lenz-vector [4]. The relevant connected component SO(4) is the same as in the case when Pauli estimated the Hydrogen spectrum before the Schroedinger equation was formulated.

The spectrum of SO(4) is well known. In addition to j comes the principal quantum number n. D shall denote the properly normalized value of the

Runge-Lenz vector which directly can be regarded as the eccentricity of the orbit (multiplied by Planck's constant). I shall not write out the commutation relations, only the expression for the relevant Casimir operator which is $C^2 = J^2 + D^2$ with eigenvalues $(n^2 - 1)\hbar^2$, where n is a positive integer. Finally $\frac{D^2+I}{\hbar^2}$ is identified with $\frac{1}{(G\omega m)^2}$, arriving at

$$n^2 - j(j+1) = \left(\frac{1}{G\omega m}\right)^2 \ . \tag{2}$$

A deeper analysis shows that j always takes values in the integers despite of the fact that SU(2) has half integer representations as well (see, for example [5]).

If the relations are rearranged in terms of the parameters, the result is $\left(\frac{m}{m_{pl}}\right)^4 = \frac{j(j+1)}{n^2-j(j+1)}$ and $\left(\frac{T}{l_{pl}}\right)^4 = j(j+1)[n^2 - j(j+1)]$, where $T = \frac{1}{2\pi\omega}$ shall symbolize the duration of one oscillation and the denominators are the Planckian units. As experienced, times are bound from zero, while masses can be larger or smaller than the Planck mass. These relations are quite satisfactory since the Planck mass and length do appear in the most fundamental relations without detour via $\hbar$ and G [6].

As a result, a 1-dimensional harmonic oscillator can be regarded as a "Hydrogen Atom" in phase space. This meets the point insofar as it sheds new light on the so called vacuum energy. Proposing an accordingly adapted theory of gravitation needs more preparatory work and shall be done elswhere. With the involvement of Newton's constant, the above quantizes spacetime and energymomentum quantities, however it yet leaves open the relation to the gravitational field. Nowhere appear the absolute scales corresponding to the Bohr radius and the Rydberg energy involving the universal dimensionless coupling constant of gravity.

References

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