R^0 Gravity

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Abstract. It is argued that the underlying symmetry of gravitation is much like that of special relativity. That is, the Planck length mediates a symmetry between spacetime and field degrees of freedom in a flat background space, just as the speed of light in special relativity mediates a symmetry between space and time. The action of this theory appears as the 4-volume of an embedded manifold in the background space.

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Cosmological data point to an "admittedly weird composition of the universe" [1, 2]. At the same time, the problems of quantum gravity [3] and interpretation of quantum physics [4] remain open. Recent evidence indicates that general relativity needs some adaptation, even in its classical segments [5, 6, 7, 8]. If true, this would destroy all its majesty, while it is unclear which set of ground rules alternatives to general relativity should satisfy [9]. The numerous "darknesses" of the universe may well have their source in the strange symmetries underlying gravitation and quantum theory, which stand in sharp contrast to the transparent symmetry of special relativity's line element. This is especially intriguing if we consider evidence that all physical concepts and even pure numbers rest on the same footing [10, 11, 12]. Could the introduction of symmetries similar to that of special relativity bring the desired leap of understanding?

The focus of this note is on gravitation, where currently Newton's constant appears in the identity of the Einstein tensor with the averaged stress-energy tensor, but not in a line element. The alternative proposed here is that spacetime and the gravitational field span a space equipped with an invariant line element involving these both concepts. The space described by this metric lies beyond any gravitational structures, so it is flat; it acts as a non-dynamic and eternal background space. A mathematical theorem states that a flat embedding space exists (!) for any curved manifold of whatever physical meaning [13]. It is also known that a space is intrinsically flat iff there exists one coordinate system which has direct meaning of distances or angles [14].

The background-dependence may be a source of immediate scepticism. However, it opens the door to sanative consequences. The Lagrangian will be expressed purely in terms of positions in the flat background space. The metric on any embedded manifold can then be deduced by virtue of Gauss' theorema egregium, but is not allowed to fluctuate independently. Consequently, to avoid high-order derivatives, apart from an overall multiplicative constant the field action is simply the induced volume of the embedded manifold. Since the curvature scalar R does not appear, this theory might be called "R^0 gravity".

The resulting model has concepts in common with string and brane theories [15, 16], although their roles are mutated to fit the context. The number of degrees of freedom associated with the gravitational field, as well as their coupling to the sources, are not determined a priori. The aim of this note is to demonstrate the principle of this class of theories using the most convincing example of a 4-component field with vector like coupling. The field degrees of freedom shall be denoted A, and be pure (dimensionless) numbers. This leads to the following coordinate representation of the line element of flat background space:

$$\mathrm{d}s^2 = \sum_{a=1}^{8} \mathrm{d}\Xi_a \mathrm{d}\Xi^a \quad = \quad \sum_{\mu=0}^{3} \mathrm{d}x_\mu \mathrm{d}x^\mu - \ell^2 \sum_{\nu=0}^{3} \mathrm{d}A_\nu \mathrm{d}A^\nu \tag{1}$$

The greek indices indicate the familiar Minkowski metric of flat, four-dimensional spacetime, with the velocity of light set to unity. The differing signs turn out to be necessary to achieve reasonable solutions, resulting in an embedding space of maximum symmetry with metric signature (++++ - - - -). ℓ is a conversion factor with dimension of length, for which the Planck length—up to factors of order unity—is the only serious candidate. The involvement of Planck's constant sheds new light on gravitation and all related concepts.

Four embedding equations are required to define a four-dimensional "field manifold" $\mathcal{M}$, which will be a subspace of this flat space. The metric signature of $\mathcal{M}$ is (+ - - -), where the overall sign is a convention. The induced volume is

$$V_{\mathcal{M}} = \int \sqrt{\left| \det \frac{\partial \Xi^a}{\partial \xi^\mu} \frac{\partial \Xi_a}{\partial \xi^\nu} \right|} \, \mathrm{d}^4\xi \ , \tag{2}$$

where ξ symbolizes a degree of freedom of the field manifold. In the noncosmological approximation, the ξ can be identified with the x in (1).

The field manifold is not a solution to the field equations of General Relativity. Neither is the induced metric on the field manifold equivalent to the gravitational field. Rather, apart from cosmological curvature, our observable spacetime can be identified with the flat subspace parametrized by the x above. The A play the role of a gravitational field, in that the action is varied in terms of these variables. What one sees as the worldsheets of sources are in fact singularities of $\mathcal{M}$ projected onto our spacetime x.

The continuous symmetries present in the action (2) are the generalized Poincaré symmetry of the eight-dimensional embedding space, and the diffeomorphism invariance of $\mathcal{M}$. This fits with gravitation. A field action which has similarities to a vector theory, arises as an approximation to (2), just as classical mechanics arises from special relativity. After renormalization by subtracting the four-volume of flat spacetime, one finds that this approximate action differs from that of electrodynamics. In particular,

Table 1. Correspondence between quantities in special relativity and R^0 gravity. The greek indices follow the Minkowski metric in four dimensions.

Special relativity	R^0 gravity
Time t	Spacetime x^μ (flat apart from cosmological curvature)
Spatial position $\vec{x}$	Gravitational field A^ν
Velocity of light c	Inverse fundamental (Planck) length ℓ^{-1}
Flat spacetime	Flat embedding space
Trajectory $\vec{x} = \vec{x}(t)$	Field manifold $A^\nu = A^\nu(x^\mu)$
Proper length τ of trajectory	4-Volume $V_{\mathcal{M}}$ of field manifold
Velocity vector	Induced metric tensor on the field manifold
Mass	$\frac{\text{“Cosmological'' constant}}{\text{Newton's constant}}$
$\mathrm{d}\tau \approx (1 - \frac{\dot{\vec{x}}^2}{2c^2})\mathrm{d}t$	$\mathrm{d}V_{\mathcal{M}} \approx (1 - \frac{\ell^2}{2}\partial_\mu A_\nu \partial^\mu A^\nu)\mathrm{d}^4x$

it lacks the divergence (gauge) term. Some correspondences of this theory with the case of special relativity are listed in Table 1.

The fundamental static (in the sense that only one field component does not vanish) solutions to the action principle for a generalized number of dimensions are denoted $\phi(r)$, where ϕ is the field. It depends only on the radial component r in fully Euclidean (here called space-like) space, which in addition to r is parameterized by n angles. Hence, the volume of the $n+1$-dimensional field manifold $\tilde{\mathcal{M}}$ is $V_{\tilde{\mathcal{M}}} \propto \int \sqrt{\left|1 \mp (\frac{\ell\,\mathrm{d}\phi}{\mathrm{d}r})^2\right|} r^n \mathrm{d}r$. The sign in front of the squared field derivative depends on whether the field is time-like or space-like. The first integration of the field equation yields

$$\left(\frac{\ell\,\mathrm{d}\phi}{\mathrm{d}r}\right)^2 = \frac{1}{1 - \left(\frac{r}{r_{max}}\right)^{2n}} \quad , \quad \left(\frac{\ell\,\mathrm{d}\phi}{\mathrm{d}r}\right)^2 = \frac{1}{\left(\frac{r}{r_{min}}\right)^{2n} - 1} \tag{3}$$

for both time-like and space-like ϕ, where the two solutions correspond to manifolds of maximum and minimum volume respectively. r_{max} and r_{min} are constants of integration. These values represent branch points, where the derivative of the field is infinite. The n-sphere where this singularity is located shall be called the “terminator”. Figure 1 gives a sketch of these solutions.

The source terms describing the divergence of the field are proportional to the terminator n-volume in both cases. For the time-like field, the source is located at the “Big Bang”, or $r = 0$.

For the space-like field, it is sensible to locate the divergence at the terminator. This yields a second $n+1$-volume in addition to those of the respective field manifold, as is indicated in Figure (1) by the vertical line. These two parts of the manifold carry opposite signs, however, in analogy to the familiar relation $\triangle\left(+\frac{1}{r}\right) = -4\pi\delta(\vec{x})$. If the volume of the field manifold is counted positive, then the source term is proportional to $-|\phi(r_{min}|$ irrespective of the sign chosen for ϕ, and vice versa. $\phi(r_{min})$ is the value at the terminator if approached from outside, starting from $\phi(\infty) = 0$. The additive constant of integration will be addressed later.

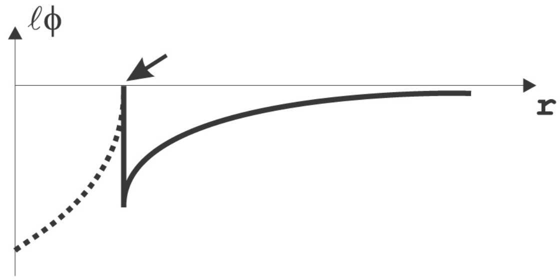

Figure 1. Fundamental solution for the time-like (dotted line) and space-like (full line) fields, for a specific choice of their respective branches and second constants of integration. The point marked by the arrow, factually an n-sphere, is the terminator. The vertical line symbolizes a source term for the space-like field, contributing with opposite sign. The line element in this plane is $\mathrm{d}s^2 = -\mathrm{d}r^2 \pm \ell^2 \mathrm{d}\phi^2$, where the overall sign is a convention and the $\pm$ refers to time-like and space-like solutions respectively. At $r = 0$ the slope is 45 degrees, while at the terminator it becomes infinite. $\phi \asymp \frac{const}{r^{n-1}}$ for $n > 1$.

These fundamental solutions have two especially important configurations. The first is the time-like field for $n = 3$, which describes an idealized universe where space-like sections at a given cosmological time are 3-spheres. In this case, however, the grouping of degrees of freedom specified in (1) is not appropriate. Rather, time itself is a field extending over the four-dimensional, fully Euclidean subspace of the embedding space. This field manifold is identical with curved spacetime in the approximation of spatial homogeneity and isotropy. All quantities have the dimension of length and consequently the Planck length plays no relevant role. The properties and relevance of this or other cosmological solutions shall be discussed elsewhere. Another point I shall not discuss here is how to unify the cosmological solution with the noncosmological approximation. The focus is the noncosmological approximation alone, based on the grouping of coordinates in (1).

The second important case involves a massive, non-rotating point source described in its own proper rest frame. This static configuration is thus translation invariant along x^0, and any other time-like degrees of freedom vanish. In this scenario x^1, x^2 and x^3 build up a basic Euclidean space which is mapped into A^0. The corresponding solution is the space-like field with $n = 2$, where $\phi \equiv A^0$ can be asymptotically identified with the Newtonian potential. Here $r_{min} = \sqrt{\ell G M}$, where G is Newton's constant and M is the mass of the point source. Furthermore, $\ell|\phi(r_{min})| = b r_{min}$. In this expression, $b = \int_1^\infty \frac{\mathrm{d}x}{\sqrt{x^4-1}} = 1,31\ldots$. The free particle term inducing an attractive force can be obtained from the substitution $|\phi| \to |\phi| - 1$, which can be associated with the second constant of integration. As a function of M, $|\phi(r_{min})| - 1$ changes sign at the Planck mass, if ℓ is b^2 times the Planck length.

Including the x_0 coordinate, the field manifold and source terms are both 4-dimensional, which can be generalized to include an arbitrary number of sources. This makes the total action $S \propto V_{\mathcal{M}} + \sum_n V_n$, where the sum is over all sources, and the relative signs were discussed earlier. Any terminator must be locally orthogonal to the spacetime trajectories of its points, since tangential motions would leave the terminator unchanged and thus can be considered unphysical in the context of the action. Since the field is orthogonal to space-time, the "vector" coupling term $\pm\ell A_\mu dx^\mu$ is actually a sum over infinitesimal two-dimensional subvolumes of the embedding space orthogonal to the terminator. The value of this sum is invariant under coupled rotations in the x and the A subpsace.

Despite this specific symmetry, there are reasons to believe that this is an appropriate source term. Mass makes sense as a conserved quantity at the level of individual sources. Its non-conservation originates from the interaction of worldsheets. Each source of nonvanishing mass contributes

$$V_n = \ell \int (u_\mu \pm A_\mu) \mathrm{d}x^\mu \mathrm{d}\mathcal{V}_n^{termi} , \tag{4}$$

where $\mathcal{V}_n^{termi}$ is the 2-volume of the corresponding terminator—this is a space-like submanifold of spacetime with topology of a sphere where the singularity of the field is located. Mass does not appear explicitely. u is the relativistic four-velocity of any point at any terminator, and is derived from the coordinate velocity $v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}x^0} = (1, \vec{v})$. The $\pm$ indicates the invariance of the dynamics under a global choice of this sign. The unphysical infinite self-force on the terminator can be avoided if a source-free field is defined on the region inside the terminator and is used for the purpose of calculating forces. The actual values of the field components at the terminator serve as "Dirichlet" boundary conditions for this source-free field.

Since the free particle term is based on the square of the four-velocity, the above expression covers massless sources as well. As usual, u_μ has to be replaced by v_μ with $v_\mu v^\mu = 0$. The terminator of a single source then reduces to a circle whose diameter is proportional to the energy, in close analogy with the Aichelburg-Sexl shockwave [17]. This implies the substitution $\mathrm{d}\mathcal{V}_{(2)}^{termi} \to \ell \cdot \mathrm{d}\mathcal{V}_{(1)}^{termi}$, where the subscript now indicates the dimensionality.

In the Newtonian approximation, the terminator area is $4\pi\ell GM$ and $\vec{v}$ is constant over the terminator. The free particle term located on the particle world line becomes $V = 4\pi\ell^2 GM \int \sqrt{1-\vec{v}^2}\mathrm{d}x^0$, which is further reduced to $-4\pi\ell^2 GM \int \frac{\vec{v}^2}{2}\mathrm{d}x^0$. The volume of the field manifold is approximately $\int \sqrt{1+(\ell\nabla A^0)^2}\mathrm{d}^4x$, which reduces to $\ell^2 \int \frac{(\nabla A^0)^2}{2}\mathrm{d}^4x$. The force is mediated by the coupling term $\pm 4\pi\ell^2 GM \int A^0 \mathrm{d}x^0$.

If the proportionality between the four-volume and the action is expressed in terms of the "cosmological constant" as $dS = \frac{\Lambda}{G}dV$, then the Newtonian limit implies $\Lambda = -\ell^{-2}$. While the absolute value appears as plausible, the sign is questionable, since the static field manifold is a minimum surface. The discrepancy can be resolved from the observation that the homogeneous and isotropic universe is a saddle point of the action. This suggests a purely imaginary value, which is possible since Λ does not enter any equation of motion. There is an explanation for this result, where the Lagrangian and consequently the Hamiltonian remain real. The metric determinant of 4-dimensional spacetime as well as of $\mathcal{M}$ is always negative, hence $\sqrt{\det g} = \pm\imath \sqrt[+]{|\det g|}$, where g is the relevant metric.

Acknowledgments

In our actual universe the symmetry in (1) is broken, as reflected by (4). We always see trajectories in spacetime, never in the field (or in energymomentum, as my academic teacher Kurt Baumann emphasized in the context of interpretation of quantum physics 25 years ago).

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